Long waves on a rotating earth in the presence of a semi-infinite barrier

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Summary

In this paper the problem is considered of long gravity waves approaching a semi-infinite barrier which extends parallel to the wave crests, the whole system being in rotation. It is well known that, when the rotation is zero, there is a 'shadow region' behind the barrier in which the disturbance diminishes rapidly with distance from the edge. However, it is shown that the rotation gives rise to an additional wave in the shadow region. The crests of this wave are at right-angles to the incident wave, and it travels along the barrier without attenuation in that direction. The amplitude falls off exponentially with distance from the barrier, as in a Kelvin wave. The amplitude at the barrier may exceed that of the incident waves.

The problem arises in connexion with the propagation of tides and storm surges in the ocean.

1. Introduction

The work described in this paper arises from certain aspects of an investigation into the origin of storm surges. A particular example of such a surge is the raising in sea-level which occurred progressively round the North Sea and produced such serious flooding in February 1953, but it is known that many surges pass unnoticed because they do not result in serious destruction (Corkan 1948, Rossiter 1954). Some may be generated by atmospheric disturbances over the North Sea itself, but others appear to originate outside the sea and then propagate into and around it in an anti-clockwise direction as free waves on the rotating earth.

Here we are concerned with a possible mechanism for the propagation of these free waves from a region to the west and north-west of the British Isles into the North Sea. However, the results may have more general applications in oceanography and meteorology.

The particle velocity associated with long waves on a rotating earth is itself rotatory when conditions are uniform along the wave crests (Proudman 1953, p. 262). If such a system of simple harmonic plane waves is incident normally on a semi-infinite barrier, one might expect that after the waves have passed the barrier their transverse velocity components act as a source for waves propagating into the region behind the barrier. In the rather similar problem in acoustics, there is no such source and this region is 'shadowed' from the incident waves, the only disturbance being due to diffraction effects arising at the edge of the barrier and dying out with distance away from it (Lamb 1932, p. 538).

In this paper, we investigate the effect of the rotation on the disturbance behind the barrier. It is found that there is indeed a system of progressive waves travelling along the barrier when it lies in the right half of the plane, as in figure 1 (the sense of rotation is assumed to be anti-clockwise as in the northern hemisphere). The crest height of these secondary waves is not uniform but decreases exponentially away from the barrier as in waves of Kelvin type (Proudman 1953, p. 253). The amplitude of the secondary waves is not attenuated with distance along the barrier. We calculate the amplitude of the waves at infinity, and show that for a certain range of frequencies they may exceed the amplitude of the incident waves. The disturbance along a path extending from the edge of the barrier in any other direction dies out rapidly with distance.

2. The equations of motion

In this paper, we assume, as in most long-wave theory, that the vertical acceleration is small (Lamb 1932, p. 254). The equation of motion in the vertical direction then reduces to the hydrostatic equation $p = g\rho_0(z+\zeta)$, where p is the pressure, z is the depth below the mean free surface, ζ is the elevation of the free surface above its mean level, and ρ_0 is the density.

The horizontal equations of motion become, on substitution of the hydrostatic equation and neglect of the non-linear and viscous terms (Proudman 1953, p. 220),

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - \gamma v &= -g\frac{\partial \zeta}{\partial x}, \\ \frac{\partial v}{\partial t} + \gamma u &= -g\frac{\partial \zeta}{\partial y}, \end{aligned}\right\} \tag{1}$$

where u, v are the components of velocity in the horizontal (x, y)-plane, and are functions of x, y, t only. The Coriolis parameter γ is equal to $2\omega \sin\phi$, where ω is the angular velocity of the earth and ϕ is the north latitude.

The equation of continuity is (Proudman 1953, p. 220)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{1}{h}\frac{\partial \zeta}{\partial t}, \tag{2}$$

where h is the mean depth of water, which is assumed constant and large compared with ζ.

From these equations, a differential equation for ζ can be derived:

$$\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} - \frac{1}{gh}\left(\frac{\partial^2 \zeta}{\partial t^2} + \gamma^2 \zeta\right) = 0. \tag{3}$$

If ζ is assumed to have a simple harmonic time factor $\exp(-i\sigma t)$, equation (3) becomes

$$\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}+k^2\zeta=0, \tag{4}$$

where $$k^2=(\sigma^2-\gamma^2)/gh,$$

and the elevation ζ is now to be taken as a function of x, y only. Throughout the paper we assume $\sigma>\gamma$.

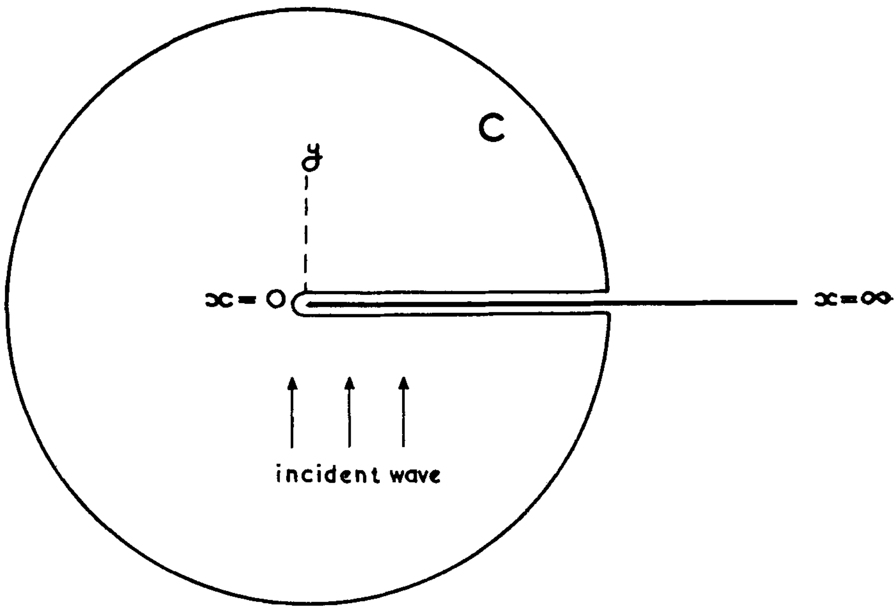

Figure 1.

The conditions of the problem are that waves of the form $a\exp[i(ky-\sigma t)]$ are incident, from $y=-\infty$, on a semi-infinite barrier placed at $y=0$, $x>0$ (see figure 1). The boundary condition on the barrier is $v=0$. To satisfy the radiation condition at infinity, it is easiest to use Copson's condition that k has a small positive imaginary part, and that the secondary waves due to the presence of the barrier tend to zero as $r\to\infty$ where $r=(x^2+y^2)^{1/2}$ (Baker & Copson 1950, p. 154). When the solution is completed, we may let the imaginary part of k tend to zero. This, or an equivalent condition, is usually assumed in practice, although it appears never to have been rigorously justified in the case of a boundary extending to infinity (Peters & Stoker 1954).

By substitution of the boundary condition on v into equations (1), the boundary condition on ζ is found to be

$$\frac{\partial\zeta}{\partial y}=\frac{i\gamma}{\sigma}\frac{\partial\zeta}{\partial x}=ip\frac{\partial\zeta}{\partial x}, \tag{5}$$

where $p=\gamma/\sigma<1$.

3. The integral equation

We follow the method used by Karp (1950) and Copson (1946) in deriving an integral equation which, when solved, enables ζ to be found. The elevation $\zeta(x,y)$ in the interior of a contour C is first expressed in terms of a Green's function and the boundary values of $\zeta(x,y)$ and $\partial\zeta(x,y)/\partial n$, where $\partial/\partial n$ denotes differentiation along the outward normal to the boundary. Thus,

$$\zeta(x,y)=\int_C\left\{G(x,y\,;x_0,y_0)\frac{\partial}{\partial n_0}\zeta(x_0,y_0)-\zeta(x_0,y_0)\frac{\partial}{\partial n_0}G(x,y\,;x_0,y_0)\right\}ds_0, \quad (6)$$

where ds_0 is the element of arc length and $G(x,y\,;x_0,y_0)$ satisfies

$$\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2}+k^2G=-\delta(x-x_0)\,\delta(y-y_0), \quad (7)$$

$\delta(x)$ being the Dirac delta function. The Green's function thus represents the effect of a source at $x=x_0$, $y=y_0$.

We choose the free-space Green's function

$$G(x,y\,;x_0,y_0)=\tfrac{1}{4}iH_0^{(1)}[k\{(x-x_0)^2+(y-y_0)^2\}^{1/2}], \quad (8)$$

where $H_0^{(1)}$ is the Hankel function of the first kind (Watson 1944, p. 73). The contour C is a large circle about the origin whose radius ultimately tends to infinity, but it is indented along the positive x-axis to exclude the barrier (figure 1). Then

$$\zeta(x,y)=\int_0^r\left([\zeta]\frac{\partial G}{\partial y_0}-G\frac{\partial}{\partial y_0}[\zeta]\right)_{y_0=0}dx_0+\int_{C_1}\left(G\frac{\partial \zeta}{\partial n_0}-\zeta\frac{\partial G}{\partial n_0}\right)ds_0, \quad (9)$$

where $[\zeta]=\zeta_+(x,0)-\zeta_-(x,0)$, the difference between ζ on the two sides of the barrier, and C_1 is the circular part of C. If we now take ζ' to represent the secondary waves arising from the incidence of the primary waves $a\exp(iky)$ on the barrier, equation (9) may be written as

$$\zeta(x,y)=\int_0^r\left([\zeta]\frac{\partial G}{\partial y_0}-G\frac{\partial}{\partial y_0}[\zeta]\right)_{y_0=0}dx_0+\int_{C_1}\left(G\frac{\partial \zeta'}{\partial n_0}-\zeta'\frac{\partial G}{\partial n_0}\right)ds_0+a\exp(iky).$$

Finally, by the radiation condition, the integral round C_1 tends to zero as the radius of the circle tends to ∞, and

$$\zeta(x,y)=\int_0^\infty\left([\zeta]\frac{\partial G}{\partial y_0}-G\frac{\partial}{\partial y_0}[\zeta]\right)_{y_0=0}dx_0+a\exp(iky). \quad (10)$$

The boundary condition on ζ (and therefore on $[\zeta]$) is now substituted from equation (5) into equation (10); and, after an integration by parts, it follows that

$$\zeta(x,y)=\int_0^\infty\left\{[\zeta]\left(\frac{\partial G}{\partial y_0}+ip\frac{\partial G}{\partial x_0}\right)\right\}_{y_0=0}dx_0+a\exp(iky)-\Big[(ipG[\zeta])_{y_0=0}\Big]_{x_0=0}^{x_0=\infty},$$

$$=\int_0^\infty\left\{[\zeta]\left(\frac{\partial G}{\partial y_0}+ip\frac{\partial G}{\partial x_0}\right)\right\}_{y_0=0}dx_0+a\exp(iky), \quad (11)$$

if it is assumed that $[\zeta]$ is bounded and that it is zero at the origin. This imposition of a specific behaviour on ζ as $x\to 0$ is necessary in this type of problem, and a condition of continuity at the origin would seem to be the most appropriate in the present case (Karp 1950).

When the operator $(\partial/\partial y - ip\partial/\partial x)$ is applied to equation (11) on $y=0$, it follows from the boundary condition, equation (5), that

$$0 = aik + \int_0^\infty \left\{ [\zeta] \left(\frac{\partial}{\partial y} - ip\frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial y_0} + ip\frac{\partial}{\partial x_0} \right) G \right\}_{y=y_0=0} dx_0, \tag{12}$$

which is valid only for $x>0$.

4. The solution of the integral equation

Equation (12) is solved by the Wiener–Hopf method (Titchmarsh 1937, p. 339), of which only the outline is given below.

Let

$$\left.\begin{aligned} f(x) &= 0 \quad & x<0, \\ &= [\zeta] \quad & x>0, \\ g(x) &= \text{a function defined by (14)} \quad & x<0, \\ &= 0 \quad & x>0, \\ q(x) &= 0 \quad & x<0, \\ &= aik \quad & x>0, \\ l(x-x_0) &= \left\{ \left(\frac{\partial}{\partial y} - ip\frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial y_0} + ip\frac{\partial}{\partial x_0} \right) G \right\}_{y=y_0=0}. \end{aligned}\right\} \tag{13}$$

Then equation (12) may be rewritten as

$$g(x) = q(x) + \int_{-\infty}^{\infty} f(x_0)\, l(x-x_0)\, dx_0, \tag{14}$$

which now holds for all x, and defines $g(x)$ for $x<0$.

Equation (14) may be solved by taking the Fourier transform of both sides of the equation. We then find that

$$G(\alpha) = Q(\alpha) + L(\alpha)F(\alpha), \tag{15}$$

where $G(\alpha)$, $Q(\alpha)$, $L(\alpha)$, $F(\alpha)$ are the Fourier transforms of $g(x)$, $q(x)$, $l(x)$, $f(x)$ respectively. Now,

$$Q(\alpha) = \int_{-\infty}^{\infty} q(x)\exp(-i\alpha x)\, dx = \int_0^\infty aik\exp(-i\alpha x)\, dx = \frac{ak}{\alpha},$$

and $Q(\alpha)$ is regular for $\mathscr{I}\{\alpha\}<0$. Similarly, we have (Erdelyi 1955)

$$L(\alpha) = \frac{i\{k^2 - \alpha^2(1-p^2)\}}{2(k^2-\alpha^2)^{1/2}},$$

and $L(\alpha)$ is regular for $|\mathscr{I}\{\alpha\}| < \mathscr{I}\{k\}$. Next, we assume that $f(x) = O(e^{ik_1x})$, with $\mathscr{I}\{k_1\} \geqslant 0$ as $x \to +\infty$, so that $f(x)$ is bounded at $+\infty$. It then follows that $F(\alpha)$ is regular and bounded for $\mathscr{I}\{\alpha\}<0$. Similarly, it follows from equation (14) that $|g(x)| = O\{\exp(-\mathscr{I}\{k\}|x|)\}$ as $x \to -\infty$, and therefore that $G(\alpha)$ is regular and bounded for $\mathscr{I}\{\alpha\} > -\mathscr{I}\{k\}$. The domains of regularity of the various transforms in the complex α-plane are shown in figure 2.

Equation (15) now becomes

$$G(\alpha)=\frac{ak}{\alpha}+\frac{i\{k^2-\alpha^2(1-p^2)\}}{2(k^2-\alpha^2)^{1/2}}F(\alpha). \tag{16}$$

The object now is to factorize $L(\alpha)$ into two parts $L_+(\alpha)$ and $L_-(\alpha)$, one regular in an upper half-plane of the complex variable α, the other in the lower half-plane, and then to express equation (16) as the equality of two

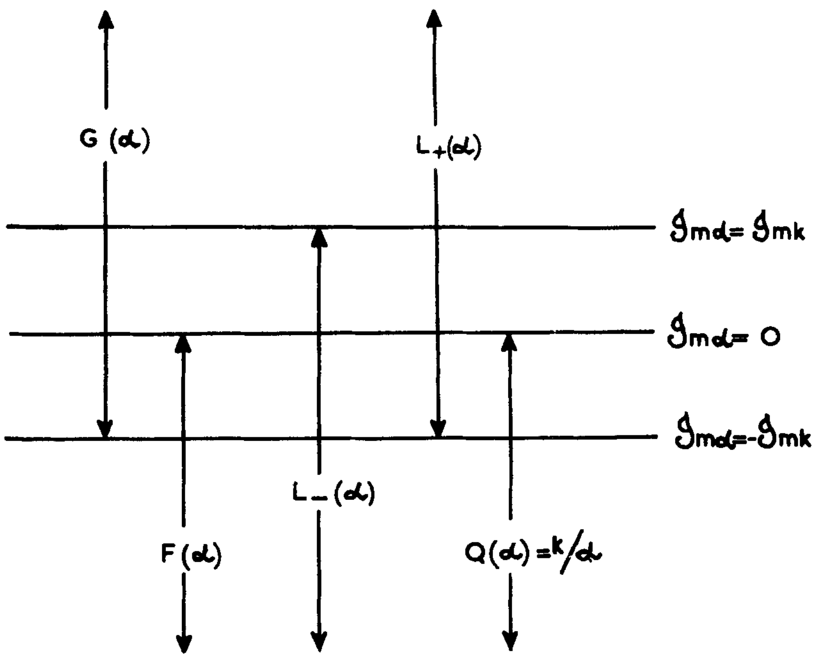

Figure 2. Domains of regularity of the transforms.

functions that are regular in two half-planes which have a common strip of regularity. The resulting equation is

$$\frac{G(\alpha)(k+\alpha)^{1/2}}{k+\alpha(1-p^2)^{1/2}}-\frac{ak^{1/2}}{\alpha}\left\{\frac{(k+\alpha)^{1/2}}{k+\alpha(1-p^2)^{1/2}}-\frac{1}{k^{1/2}}\right\}=\frac{ak^{1/2}}{\alpha}+\frac{i\{k-\alpha(1-p^2)^{1/2}\}}{2(k-\alpha)^{1/2}}F(\alpha). \tag{17}$$

The left-hand side of equation (17) is regular for $\mathscr{I}\{\alpha\}>-\mathscr{I}\{k\}$, and the right-hand side for $\mathscr{I}\{\alpha\}<0$. Therefore, as both sides are regular in the strip $0>\mathscr{I}\{\alpha\}>-\mathscr{I}\{k\}$, they define a function $E(\alpha)$ which is regular over the entire α-plane by analytic continuation.

It follows from equation (17) that, as $G(\alpha)$, $F(\alpha)$ are bounded in their respective half planes, $E(\alpha)$ is $O(|\alpha|^{1/2})$ for the lower half plane and $O(|\alpha|^{-1/2})$ for the upper half plane. Thus $E(\alpha)$ is $O(|\alpha|^{1/2})$ as $|\alpha|\to\infty$. It then follows, from an extension to Liouville's theorem, that $E(\alpha)$ is a polynomial of order less than 1/2 and therefore a constant. This constant is zero because $E(\alpha) = O(|\alpha|^{-1/2})$ as $|\alpha| \to \infty$ in the upper half plane (Karp 1950). Therefore,

$$\frac{i\{k-\alpha(1-p^2)^{1/2}\}}{2(k-\alpha)^{1/2}} F(\alpha) + \frac{ak^{1/2}}{a} = 0,$$

so that

$$F(\alpha) = \frac{2iak^{1/2}(k-\alpha)^{1/2}}{\alpha\{k-\alpha(1-p^2)^{1/2}\}}. \tag{18}$$

Although the explicit form of $f(x)$ will not be required, it could now be calculated by means of the Fourier reciprocal theorem:

$$f(x) = \frac{1}{2\pi}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} F(\alpha)\exp(i\alpha x)\,d\alpha \quad 0<\epsilon<\mathscr{I}\{k\}. \tag{19}$$

5. The solution $\zeta(x, y)$

From equation (11), we have

$$\begin{aligned}\zeta(x,y) &= \int_0^\infty \left\{[\zeta]\left(\frac{\partial G}{\partial y_0} + ip\frac{\partial G}{\partial x_0}\right)\right\}_{y_0=0} dx_0 + a\exp(iky),\\ &= \int_{-\infty}^{\infty} f(x_0)\, m(x-x_0)\, dx_0 + a\exp(iky),\end{aligned} \tag{20}$$

where

$$m(x) = \frac{ik}{4}\frac{(y+ipx)}{(x^2+y^2)^{1/2}} H_1^{(1)}[k(x^2+y^2)^{1/2}], \tag{21}$$

and its transform is (Erdleyi 1955)

$$M(\alpha) = \frac{1}{2}\left(\frac{y}{|y|} + \frac{i\alpha p}{(k^2-\alpha^2)^{1/2}}\right)\exp\{i|y|(k^2-\alpha^2)^{1/2}\}, \tag{22}$$

which is regular in the strip $|\mathscr{I}\{\alpha\}| < \mathscr{I}\{k\}$. The Fourier convolution theorem states that

$$\int_{-\infty}^{\infty} f(x_0)\, m(x-x_0)\, dx_0 = \frac{1}{2\pi}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} F(\alpha)\, M(\alpha)\exp(i\alpha x)\, d\alpha.$$

By this theorem, equation (20) may be written as

$$\begin{aligned}\zeta(x,y) = \frac{1}{2\pi}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} &\frac{2iak^{1/2}(k-\alpha)^{1/2}}{\alpha(k-\alpha s)}\frac{1}{2}\left(\frac{y}{|y|} + \frac{i\alpha p}{(k^2-\alpha^2)^{1/2}}\right)\\ &\times \exp[i\{\alpha x + |y|(k^2-\alpha^2)^{1/2}\}]\, d\alpha + a\exp(iky),\end{aligned} \tag{23}$$

where $s^2 = 1-p^2$. For convenience we now split the right-hand side of equation (23) into three parts:

$$\zeta = \zeta_1 + \zeta_2 + \zeta_3,$$

where

$$\left.\begin{aligned}\zeta_1 &= a\exp(iky),\\ \zeta_2 &= \frac{1}{2\pi}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{\alpha i k^{1/2}}{\alpha(k-\alpha)^{1/2}}\frac{y}{|y|}\exp[i\{\alpha x + |y|(k^2-\alpha^2)^{1/2}\}]\, d\alpha,\\ \zeta_3 &= \frac{1}{2\pi}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} a\left\{\frac{(s-1)ik^{1/2}}{(k-\alpha)^{1/2}(k-\alpha s)}\frac{y}{|y|} - \frac{k^{1/2}p}{(k+\alpha)^{1/2}(k-\alpha s)}\right\}\\ &\qquad\times \exp[i\{\alpha x + |y|(k^2-\alpha^2)^{1/2}\}]\, d\alpha.\end{aligned}\right\} \tag{24}$$

We consider first the expression ζ_3. Copson (1946) has shown how integrals of this nature may be transformed into more recognizable forms. Let $x=\rho\cos\theta$, $y=\rho\sin\theta$, and consider the integral in the first quadrant $0\leqslant\theta\leqslant\pi/2$. Then the line of integration may be deformed to that given by

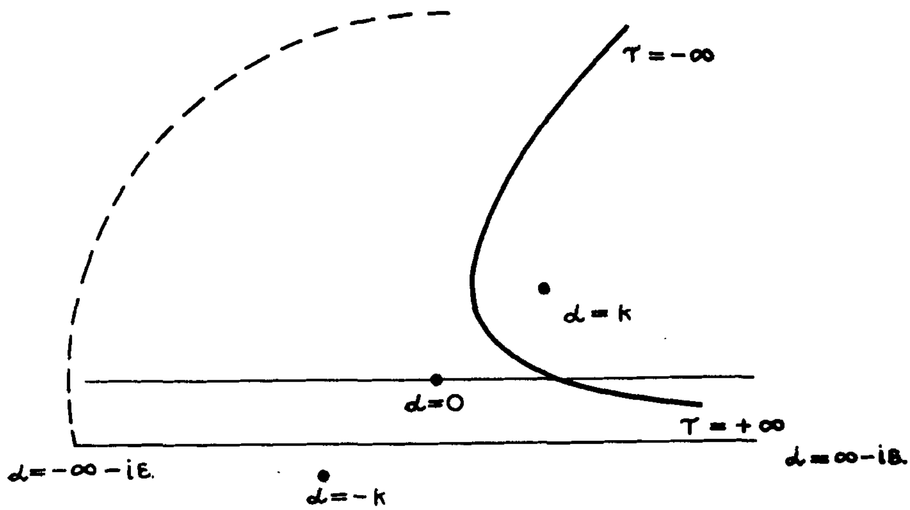

Figure 3. Paths of integration in α-plane.

$\alpha=k\cos(\theta+i\tau)$, with τ real and with the limits $-\infty$, ∞ (figure 3). After some manipulation it is found that

$$\zeta_3 = \frac{2\sqrt{2ia}}{\pi}\sinh\tfrac{1}{2}\beta\int_0^\infty \frac{\sin\frac{1}{2}(\theta-i\beta)\cosh\frac{1}{2}\tau}{\cosh\tau-\cos(\theta-i\beta)}\exp(ik\rho\cosh\tau)\,d\tau, \tag{25}$$

$$= \frac{2\sqrt{2ia}}{\pi}\sinh\tfrac{1}{2}\beta\exp\{ik\rho\cos(\theta-i\beta)\}\int_0^\infty \frac{\sin\frac{1}{2}(\theta-i\beta)\cosh\frac{1}{2}\tau}{\cosh\tau-\cos(\theta-i\beta)} \times \exp[ik\rho\{\cosh\tau-\cos(\theta-i\beta)\}]\,d\tau, \tag{26}$$

where $\cosh\beta=1/s$.

Consider now the integral

$$I(x)=\int_0^\infty \frac{\cosh\frac{1}{2}\tau\sin\frac{1}{2}\phi}{\cosh\tau-\sin\phi}\exp[ix(\cosh\tau-\cos\phi)]\,d\tau. \tag{27}$$

Then differentiation under the integral sign leads to

$$\frac{dI(x)}{dx} = \int_0^\infty i\cosh\tfrac{1}{2}\tau\sin\tfrac{1}{2}\phi\exp[ix(\cosh\tau-\cos\phi)]\,d\tau,$$

since this integral is uniformly convergent for $x>\delta$, where δ is any small positive number. Hence

$$\frac{dI(x)}{dx} = \left(\frac{\pi}{2}\right)^{1/2} i\exp(\tfrac{1}{4}i\pi)\sin\tfrac{1}{2}\phi\,\frac{\exp(2ix\sin^2\frac{1}{2}\phi)}{\alpha^{1/2}}.$$

This may be integrated to yield

$$I(x) = \pi^{1/2} \exp(\tfrac{1}{4}i\pi) \int_0^{(2x)^{\frac{1}{2}} \sinh \frac{1}{2}i\phi} \exp(-i\lambda^2)\, d\lambda + \text{constant}. \tag{28}$$

On putting $x = 0$ in equation (27), we find

$$I(0) = \tfrac{1}{2}\pi.$$

Thus, as $I(x)$ given in equation (28) is uniformly convergent for $x \geqslant 0$,

$$\begin{aligned} I(x) &= \pi^{1/2} \exp(\tfrac{1}{4}i\pi) \int_0^{(2x)^{\frac{1}{2}} \sinh \frac{1}{2}i\phi} \exp(-i\lambda^2)\, d\lambda + \tfrac{1}{2}\pi, \\ &= \pi^{1/2} \exp(\tfrac{1}{4}i\pi) \int_{-\infty}^{(2x)^{\frac{1}{2}} \sinh \frac{1}{2}i\phi} \exp(-i\lambda^2)\, d\lambda. \end{aligned} \tag{29}$$

On substituting equation (29) into equation (26), we obtain

$$\zeta_3 = -\frac{2a}{\pi^{1/2}} \left(\frac{1-s}{s}\right)^{1/2} \exp[-\tfrac{1}{4}i\pi + ik\rho \cos(\theta - i\beta)] \times \int_{-\infty}^{(2k\rho)^{\frac{1}{2}} \sinh \frac{1}{2}(\beta + i\theta)} \exp(-i\lambda^2)\, d\lambda, \tag{30}$$

or

$$\zeta_3 = -\frac{2a}{\pi^{1/2}} \left(\frac{1-s}{s}\right)^{1/2} \exp(-\tfrac{1}{4}i\pi + ik\rho) \left[\exp(iz^2) \int_{-\infty}^{z} \exp(-i\lambda^2)\, d\lambda\right], \tag{31}$$

where $z = (2k\rho)^{1/2} \sinh \frac{1}{2}(\beta + i\theta)$.

When considering ζ_3 in the other quadrants, the same type of transformation is used except that, for $x < 0$, the α-contour is doubled back round the branch line which radiates from $\alpha = -k$. The results may be expressed by the same equation (31) provided θ is allowed to range from 0 to 2π.

The transformation of ζ_2 proceeds along similar lines. Neither ζ_2 nor ζ_1 contains the rotational parameters at all. These functions may be expected to represent the usual diffraction and reflection effects of acoustics when the boundary condition is that the normal gradient of the dependent variable is zero on the barrier.

Before stating the solution, it should be mentioned that the deformation of the contour when $x > 0$ involves crossing a pole at $\alpha = 0$. This pole contributes a term which cancels with ζ_1 for $y > 0$ (in the shadow zone) and reinforces it for $y < 0$ (total reflection). When $x < 0$, the pole is not crossed.

The solution $\zeta_1 + \zeta_2$ may now be stated in the form

$$\zeta_1 + \zeta_2 = \tfrac{1}{2}a\{\exp(iky) + \exp(iky)\} - \frac{a}{\pi^{1/2}} \exp(-\tfrac{1}{4}i\pi) \Big\{ \exp(iky) \times \int_0^{(2k\rho)^{\frac{1}{2}} \sin \frac{1}{2}(\frac{1}{2}\pi + \theta)} \exp(i\lambda^2) d\lambda + \exp(iky) \int_0^{(2k\rho)^{\frac{1}{2}} \sin \frac{1}{2}(\frac{1}{2}\pi - \theta)} \exp(i\lambda^2)\, d\lambda \Big\}, \tag{32}$$

which is similar to the solution given by Lamb (1942, p. 540).

6. Discussion

The discussion will be concerned only with ζ_3, which represents the rotational effects. The part of the solution $\zeta_1+\zeta_2$ is well known and has been fully investigated in the past.

The expression in square brackets in equation (31) occurs in other diffraction problems, but the tabulation of it appears to be confined to a short table in the range 0 (·01) 1·0 for $|z|$ and 0° (1°) 45° for $\arg z$ (Clemmow & Mumford 1952). The function would be determined for all values of $\arg z$ by a table covering the range $-45° \leqslant \arg z \leqslant 45°$. To obtain a general idea of the behaviour of ζ_3 near the origin, further values of the function have been computed. In figure 4, the real and imaginary parts of ζ_3 are

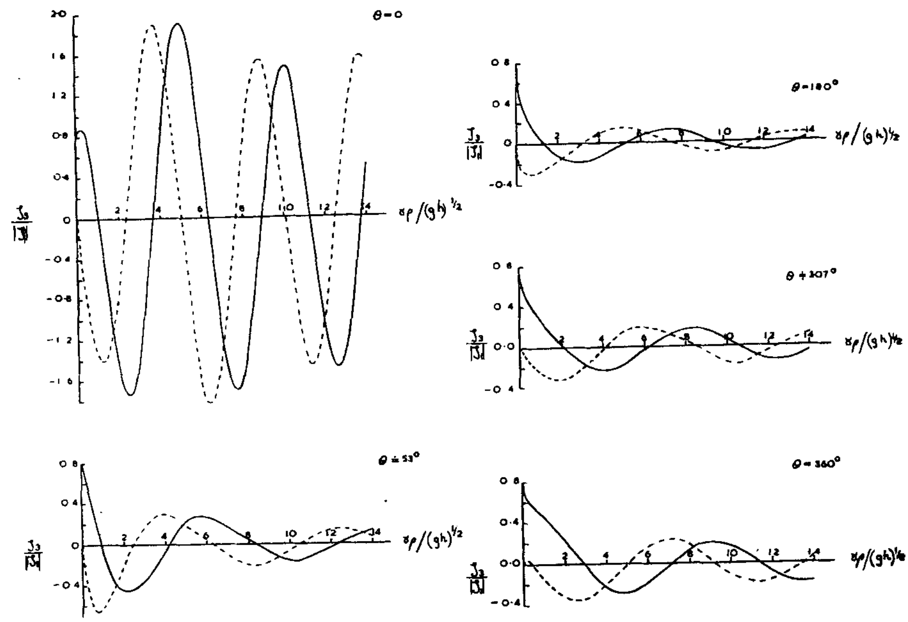

Figure 4. The real (dotted line) and imaginary (full line) parts of $\zeta_3/|\zeta_1|$ for $s=3/5$ at different angles θ to the barrier.

plotted against ρ for lines radiating from the barrier's edge at different angles to the barrier for $s=3/5$ ($p=4/5$). The amplitudes are given as the ratios of ζ_3 to $|\zeta_1|$. The horizontal scale is in units of $(gh)^{1/2}/\gamma$, which, for the latitude and depth of the North Sea, means units of 200 km approximately ($\gamma \doteqdot 10^{-4}\text{sec}^{-1}$). If the asymptotic behaviour of ζ_3 for large values of ρ is investigated, it is found that, except near the barrier for $y>0$, ζ_3 is of the order of $1/\rho^{1/2}$ for large ρ. Near the barrier, the asymptotic behaviour for $y>0$ is

$$\zeta_3 \sim 2ia\left(\frac{1-s}{s}\right)^{1/2}\exp(ikx\cosh\beta - ky\sinh\beta) + O\left(\frac{1}{\rho^{1/2}}\right)$$

$$\sim 2ia\left\{\frac{\sigma}{(\sigma^2-\gamma^2)^{1/2}}-1\right\}^{1/2}\exp\{(i\sigma x-\gamma y)/(gh)^{1/2}\} + O\left(\frac{1}{\rho^{1/2}}\right). \quad (33)$$

Thus only in the region just behind the barrier are waves propagated which do not die out away from the edge of the barrier. It follows from equation (33) that, when $p>3/5$, these secondary waves are, at large distances down the barrier, larger in amplitude than the incident waves, although they die out exponentially away from the barrier.

For waves to be propagated into the region behind it, the barrier must lie in the right half of the x-axis, as in the problem studied. If it is in the left half ($x<0$), we may expect waves progressing down the front of the barrier. This leads to the interesting possibility that, if the barrier is of finite length (an island in mid ocean), a certain amount of energy will be trapped in the form of a wave progressing round the barrier in a clockwise direction.

I am indebted to Mrs P. Edwards for the computation of the integral in equation (31).

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